Chapter 3:
Determinants and Diagonalization

Sec. 3.2:
Determinants and Matrix Inverses

Determinant Properties/Theorems

Theorem 3.2.1: Product Theorem

If A and B are $n \times n$ matrices, then $\det(AB) = \det A \det B$.

Proof:

Extended version of the product theorem:

If A_1 , A_2 , … , A_k are all $n \times n$ matrices, then

$$\det(A_1 A_2 \cdots A_{k-1} A_k) = \det(A_1) \det(A_2) \cdots \det(A_{k-1}) \det(A_k)$$

Determinant Properties/Theorems

Theorem 3.2.2

An $n \times n$ *matrix* A *is invertible if and only if* $\det A \neq 0$. *When this is the case,* $\det\left(A^{-1}\right) = \frac{1}{\det A}$

Proof:

Determinant Properties/Theorems

Theorem 3.2.3

If A is any square matrix, $\det A^T = \det A$.

Proof:

Determinant Properties/Theorems

Ex 1: If A, B, and C are $n \times n$ matrices with $|A| = 2$, $|B| = -1$, and $|C| = -3$, find

a) $|AB|$
b) $|C^T|$
c) $|A^4|$
d) $|A^{-1}|$
e) $|A^4 B^T C^{-1} A^{-1} B^2 A^{-1}|$

Determinant Properties/Theorems

Def: An $n \times n$ matrix A is called orthogonal if $A^{-1} = A^T$.

Ex 2: If A is an $n \times n$ orthogonal matrix, what are the only possible values for $\det(A)$?

The Adjoint of a Matrix and Matrix Inverses

Recall: If A is an $n \times n$ matrix…

- A_{ij} is the matrix obtained by deleting the ith row and jth column of the matrix A
- $M_{ij} \equiv \det(A_{ij})$ are called the minors of matrix A
- $c_{ij}(A) \equiv (-1)^{i+j} M_{ij}$ or $c_{ij}(A) \equiv (-1)^{i+j} \det(A_{ij})$ are called the cofactors of the matrix A

Def: If A is an $n \times n$ matrix, the adjoint of A is the transpose of the matrix of cofactors of A. (Notation: $adj(A)$)

$$adj(A) = \begin{bmatrix} c_{11}(A) & c_{12}(A) & \cdots & c_{1n}(A) \\ c_{21}(A) & c_{22}(A) & & c_{2n}(A) \\ \vdots & & \ddots & \vdots \\ c_{n1}(A) & c_{n2}(A) & \cdots & c_{nn}(A) \end{bmatrix}^T$$

The Adjoint of a Matrix and Matrix Inverses

Definition 3.3 Adjugate of a Matrix

The **adjugate**[4] *of* A*, denoted* adj (A)*, is the transpose of this cofactor matrix; in symbols,*

$$\text{adj}\,(A) = \left[c_{ij}(A)\right]^T$$

Ex 3: If $A = \begin{bmatrix} -2 & 5 & 4 \\ 1 & 3 & -1 \\ 0 & -6 & 2 \end{bmatrix}$, find

a) $|A|$

b) $adj(A)$

c) $A\ adj(A)$

d) $adj(A)\ A$

e) A^{-1}

The Adjoint of a Matrix and Matrix Inverses

Note:

It is important to note that this theorem is *not* an efficient way to find the inverse of the matrix A. For example, if A were 10×10, the calculation of $\operatorname{adj} A$ would require computing $10^2 = 100$ determinants of 9×9 matrices! On the other hand, the matrix inversion algorithm would find A^{-1} with about the same effort as finding $\det A$. Clearly, Theorem 3.2.4 is not a *practical* result: its virtue is that it gives a formula for A^{-1} that is useful for *theoretical* purposes.

The Adjoint of a Matrix and Matrix Inverses

What do you get if you multiply A with $adj(A)$?

$$A\,adj(A) = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix}, \text{ so } A^{-1} = \frac{1}{|A|}adj(A)$$

Theorem 3.2.4: Adjugate Formula

If A is any square matrix, then

$$A(\text{adj } A) = (\det A)I = (\text{adj } A)A$$

In particular, if det A $\neq$ 0, the inverse of A is given by

$$A^{-1} = \frac{1}{\det A}\,\text{adj } A$$

The Adjoint of a Matrix and Matrix Inverses

Ex 4: If $A = \begin{bmatrix} 4 & 1 & 7 \\ 2 & 0 & 9 \\ -1 & 3 & 1 \end{bmatrix}$, find the (3,2)-entry of A^{-1}

The Adjoint of a Matrix and Matrix Inverses

Ex 5: Show that if A is an $n \times n$ matrix with $n \geq 2$, then $|adj(A)| = |A|^{n-1}$.

Cramer’s Rule

- Cramer’s Rule is a cute little formula that gives you the solution to a system of n equations in n unknowns when the system has a unique solution
- The formula for Cramer’s Rule involves determinants
- Cramer’s Rule can be derived from the adjoint formula for A^{-1}

Cramer's Rule

Suppose you are trying to solve the following system of n equation in n unknowns…

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

This system can also be written as $A\boldsymbol{x} = \boldsymbol{b}$ where

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

A is the coefficient matrix

Column matrix of variables

Column matrix of constants

Cramer's Rule

System: $A\boldsymbol{x} = \boldsymbol{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $|A| = 0$, then the system will either have no solution, or infinitely many solutions.

If $|A| \neq 0$, then the system will have exactly one solutions and in this case, Cramer's Rule gives us the solution.

Cramer's Rule

Theorem 3.2.5: Cramer's Rule[5]

If A is an invertible $n \times n$ matrix, the solution to the system

$$A\mathbf{x} = \mathbf{b}$$

of n equations in the variables $x_1, x_2, \ldots, x_n$ is given by

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \cdots, \quad x_n = \frac{\det A_n}{\det A}$$

where, for each k, A_k is the matrix obtained from A by replacing column k by $\mathbf{b}$.

Ex 6: Use Cramer's Rule to solve the system of equations below…

$$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 2 \\ x_1 - 3x_2 + 6x_3 &= -7 \end{aligned}$$

What you need to know from the book

Book reading

Pages: 158-165 (only top half, do not go on to polynomial interpolation).

Problems you need to know how to do from the book

#'s 1-6, 8-21, 26-29, 32-34